POSITIVE DEFINITE COLLECTIONS OF DISKS

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Abstract. Let $Q(z,w) = -\prod_{k=1}^{n}[(z-a_k)(\bar{w}-\bar{a}_k)-R^2]$. The main result of the paper states that in the case when the nodes a_j are situated at the vertices of a regular n-gon inscribed in the unit circle, the matrix $Q(a_i,a_j)$ is positive definite if and only if $R<\rho_n$, where $z=2\rho_n^2-1$ is the smallest $\neq -1$ zero of the Jacobi polynomial $\mathcal{P}_{\nu}^{n-2\nu,-1}(z)$, $\nu=[n/2]$.

1. Introduction

Let $\mathcal{B} := \{B(a_j,R_j)\}_{1\le j\le n}$ denote the collection of open disks centered at a_j with radii $R_j>0$. The function

$$Q(z,w) = -\prod_{k=1}^{n}[(z-a_k)(\bar{w}-\bar{a}_k)-R_k^2],$$

defines the polarized equation of the union of disks in $\mathcal{B}$. Throughout this paper $Q^{\mathcal{B}}$ denotes the matrix with entries

$$Q_{ij}^{\mathcal{B}} := Q(a_i,a_j) = -\prod_{k=1}^{n}[a_{ik}\bar{a}_{jk}-R_k^2], \tag{1}$$

where

$$a_{ij} = a_i - a_j. \tag{2}$$

We will say that a collection of disks $\mathcal{B}$ is *positive* if the corresponding matrix $Q^{\mathcal{B}}$ is positive definite. Our start point is a recent result of B. Gustafsson and M. Putinar which states: *If $\mathcal{B}$ consists of disjoint disks then $\mathcal{B}$ is positive* [6, Lemma 3.1].

This result was obtained as a corollary of the general positivity property of the exponential transform for quadrature domains. We only mention that the exponential transform is regarded as a renormalized Riesz potential, and it is instrumental in recovering a measure from its moments. The above positivity phenomenon goes back to the operator theoretic origins of the exponential transform and these involve the highly sophisticated theory of the principal function of a semi-normal operator (the interested reader is referred to [4] and [5]). This is why the authors of [6] proposed a problem of finding a direct proof of the above mentioned positivity results.

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One of the interesting and intriguing aspects of the above problem is a rather unexpected interplay between geometry and analysis (the disjointness condition and the positivity of a certain matrix). Nevertheless, it turns out that in general the positivity of a collection of disks does not yield its disjointness. Namely, straightforward calculations for $n = 2$ show that matrix $Q^{\mathcal{B}}$ remains positive definite even if the discs overlap a little. It is easy to check that the positive definiteness is equivalent to the inequality

$$R_1^2 + R_2^2 < |a_1 - a_2|^2,$$

whereas the disjointness condition is expressed as

$$R_1 + R_2 < |a_1 - a_2|.$$

On the other hand, the method of [6] is completely based on the geometry of *disjoint* disks and it is no more applicable to general collections. In this connection, the main problem is to find an adequate language, geometrical or functional, for understanding of the above phenomena in the general case.

In the present paper, we completely solve this problem in the case when $\mathcal{B}$ consists of n congruent disks centered in the vertices of a regular n-gon. The main result, Theorem 1 below, states that the positivity of such a collection can be characterized in terms of the zeroes of the Jacobi polynomials.

The paper organized as follows: In Section 2 we introduce the main notation and state the main results. In Section 3 we treat the general collections. In Section 4 we establish an explicit factorization of the determinant function and reformulate the positivity problem to a problem for the zero distribution of the Jacobi polynomials. The concrete study of the zeroes is given in Section 5. In Section 6 we give the proof of Theorem 1. In the final sections we establish two-side estimates on the maximal radius.

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2. Main results

Let $a_j = \omega^j$, $j = 1, \ldots, n$, be the vertices of the regular n-gon inscribed in the unit circle, where $\omega = e^{2\pi \mathrm{i}/n}$ denotes the nth root of unity. We will denote by

$$\mathcal{B}_n(r) = \{B(\omega^j, r), \quad j = 1, \ldots, n\} \tag{3}$$

the corresponding collection consisting of n congruent disks and introduce

$$\rho_n = \sup\{\rho > 0| \quad \mathcal{B}_n(r) \text{ is positive for all } r \in (0, \rho)\}, \tag{4}$$

which we refer to as the *maximal radius* of $\mathcal{B}_n(r)$.

We recall also the definition of Bessel function of the first kind

$$J_k(x) = \left(\frac{x}{2}\right)^k \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m!\Gamma(m+k+1)}.$$

It is well-known that $J_k(z)$ has an infinite sequence of positive zeroes; we denote them $j_{k,i}$.

Theorem 1. *In the above notation, $\rho_2 = \sqrt{2}$, $\rho_3 = 1$, and for $n \geq 4$*

$$\rho_n = \sqrt{1+\mu_n},$$

where μ_n denotes the smallest $\neq -1$, zero of the hypergeometric polynomial

$$z^\nu F(-\nu, \nu - n; 1-n; -\frac{1}{z}).$$

Here F is the classical Gauss hypergeometric function and $\nu = [n/2]$ is the integer part of $n/2$. Furthermore, the following asymptotic holds

$$\lim_{n\to\infty} n\rho_n = j_{1,1}, \tag{5}$$

where $j_{1,1} = 3.831706\ldots$ is the first positive zero of the Bessel function $J_1(z)$.

The above asymptotic behavior admits a clear geometric interpretation. Namely, given a general (not necessarily symmetric) collection $\mathcal{B}$, let us define

$$\beta(\mathcal{B}) := \min_{i\neq j} \frac{R_i + R_j}{|a_i - a_j|}.$$

This quantity can be characterized as a measure of overlapping of the disks in $\mathcal{B}$ in the following sense: $\beta \leq 1$ if and only if $\mathcal{B}$ is a disjoint collection. In the symmetric case $\mathcal{B}_n(r)$ this quantity is easily found as

$$\beta(\mathcal{B}_n(r)) = \frac{r}{\sin\frac{\pi}{n}}.$$

Hence, the measures of overlapping for *positive* symmetric collections of n congruent disks lie in the following interval

$$0 < \beta(\mathcal{B}_n(r)) < \frac{\rho_n}{\sin\frac{\pi}{n}} =: \beta_n.$$

Due to (5), we have the following asymptotic behaviour for the upper bound of the previous interval

$$\beta_n \sim \frac{j_{1,1}}{\pi} = 1.219669891\ldots$$

as n goes to infinity. It is interesting to note that *asymptotically the overlapping measure stays greater than 1.*

A straightforward computation for small values of $n \geq 2$ shows that β_{2n} and β_{2n-1} are increasing subsequences. Though we are unable to prove this observation, we show in Corollary 5 below that $\beta_n > 1$ for

all $n \geq 2$. In other words, the extremal symmetric collections $\mathcal{B}_n(\rho_n)$ have non-trivial overlapping for all $n \geq 2$.

3. General collections and the maximal radius

In this section we consider the general collections $\mathcal{B} := \{B(a_j, R_j)\}_{j\leq n}$ if not stated otherwise. Such a collection is said to be *admissible* if for any k, $1 \leq k \leq n$, and any $j \neq k$

$$0 < R_k < |a_j - a_k|. \tag{6}$$

Geometrically (6) means that $a_k \notin B(a_j, R_j)$ for all $k \neq j$.

Proposition 1. *Let $\{a_j\}_{j\leq n}$ be an arbitrary collection of pairwise distinct points. Then there is an $\varepsilon > 0$ such that the collection $\{B(a_j, R_j)\}_{j\leq n}$ is positive for any choice of radii, subject to condition $0 < R_j < \varepsilon$.*

Proof. By (6) we have

$$|a_{ik}a_{jk}| > R_k^2$$

for all $k \neq i, j$. Hence for $i = j$

$$Q_{ii} = R_i^2 |\alpha_i|^2 \prod_{k\neq i} \left(1 - \frac{R_k^2}{a_{ik}\bar{a}_{ik}}\right),$$

and for $i \neq j$ by virtue of (2)

$$Q_{ij} = R_i^2 R_j^2 \frac{\alpha_i \bar{\alpha}_j}{|a_{ij}|^2} \prod_{k\neq i,j} \left(1 - \frac{R_k^2}{a_{ik}\bar{a}_{jk}}\right),$$

where

$$\alpha_i := \prod_{k=1,k\neq i}^{n} a_{ik} \neq 0.$$

In particular, $Q_{ij} \equiv 0$ if all $R_j = 0$.

Let E denote the matrix with normalized entries

$$E_{ij} = \prod_{k} \left(1 - \frac{R_k^2}{a_{ik}\bar{a}_{jk}}\right),$$

where the product is taken over all indices k such that $k \neq i, j$, and set

$$S_{ij} = \begin{cases} 1, & j = i, \\ R_i R_j / |a_{ij}|^2, & j \neq i, \end{cases}$$

so that $Q_{ij} = E_{ij} S_{ij} \cdot (R_i \alpha_i) \cdot (R_j \bar{\alpha}_j)$. Hence the quadratic form

$$\mathbf{Q}(\xi) = \sum_{i,j=1}^{n} Q_{ij} \xi_i \bar{\xi}_j$$

is equivalent (up to a linear change of variables: $\eta_i = R_i \alpha_i \xi_i$) to the form

$$\mathbf{Q}'(\eta) = \sum_{i,j=1}^{n} E_{ij} S_{ij} \eta_i \bar{\eta}_j.$$

But for the latter form we have

$$\lim_{\mathbf{R}\to 0} E_{ij}S_{ij} = I,$$

where $\mathbf{R} = (R_1, \dots, R_n)$ and I denotes the unit matrix. Hence by a continuity argument, $\mathbf{Q}'(\eta)$ is positive definite for all vectors $\mathbf{R}$ with sufficiently small norm and the desired property follows.

□

Proposition 2. *Let* $\{B(a_j, R_j)\}_{j\le n}$ *be a positive collection. Then the following assertions hold:*

(i) Any subcollection $\{B(a_i, R_i)\}_{i\in I}$ *where* $I \subset \{1, 2, \dots n\}$ *is positive.*

(ii) For $0 < r_j \le R_j$ *the new collection* $\{B(a_j, r_j)\}_{j\le n}$ *is positive.*

Proof. It suffices to prove (i) only for $I = \{1, \dots, n-1\}$. Consider the quadratic form

$$\mathbf{Q}(\xi_1, \dots, \xi_n) := \sum_{i,j=1}^{n} Q_{ij}\xi_i\bar{\xi}_j,$$

where $\|Q_{ij}\|$ is the matrix in (1), and let

$$\mathbf{Q}^I(\eta_1, \dots, \eta_{n-1}) := \sum_{i,j=1}^{n-1} Q^I_{ij}\eta_i\bar{\eta}_j, \tag{7}$$

where $\|Q^I_{ij}\|$ corresponds to the reduced system $\{B(a_i, R_i)\}_{i\in I}$. We have

$$Q^I_{ij} = -\prod_{k=1}^{n-1}[a_{ik}\bar{a}_{jk} - R_k^2],$$

where $a_{ij} = a_i - a_j$.

Since $\mathbf{Q}$ is positive definite we have

$$\mathbf{Q}(\eta_1, \dots, \eta_{n-1}, 0) = \sum_{i,j=1}^{n-1} Q_{ij}\eta_i\bar{\eta}_j > 0 \tag{8}$$

for all nontrivial vectors $(\eta_1, \dots, \eta_{n-1}) \ne 0$.

On the other hand, for $1 \le i, j \le n-1$ we have

$$Q_{ij} = -\prod_{k=1}^{n}[a_{ik}\bar{a}_{jk} - R_k^2] = (a_{in}\bar{a}_{jn} - R_n^2)Q^I_{ij}.$$

Hence substituting the last identity into (7) and using (6) yields

$$\begin{aligned}
\mathbf{Q}^I(\eta_1,\ldots,\eta_{n-1}) &= \sum_{i,j=1}^{n-1} \frac{Q_{ij}}{a_{in}\bar{a}_{jn}-R_n^2}\eta_i\bar{\eta}_j \\
&= \sum_{m=1}^{\infty}\sum_{i,j=1}^{n-1} \frac{1}{a_{in}\bar{a}_{jn}}\left(\frac{R_n^2}{a_{in}\bar{a}_{jn}}\right)^m Q_{ij}\eta_i\bar{\eta}_j \\
&= \sum_{m=1}^{\infty} R_n^{2m}\mathbf{Q}\left(\frac{\eta_1}{a_{1n}^{m+1}},\ldots,\frac{\eta_{n-1}}{a_{n-1,n}^{m+1}},0\right) \\
&\geq 0,
\end{aligned} \tag{9}$$

and the above series converges absolutely because of (6).

Taking into account (8), we see that the strict inequality in (9) holds for all $(\eta_1,\ldots,\eta_{n-1}) \neq 0$, and the first assertion of the theorem is proved.

In order to prove (ii) we assume that $\mathbf{Q}$ is a positive definite form, and let r_j be any arbitrary reals subject to condition $0 < r_j < R_j$ and denote by $\|q_{ij}\|$ the corresponding matrix. Then we have

$$q_{ij} = -\prod_{k=1}^{n}[a_{ik}\bar{a}_{jk}-r_k^2] = Q_{ij}\prod_{k=1}^{n}\frac{a_{ik}\bar{a}_{jk}-r_k^2}{a_{ik}\bar{a}_{jk}-R_k^2}. \tag{10}$$

We claim that for any k the matrix with the entries

$$\alpha_{ij} = \frac{a_{ik}\bar{a}_{jk}-r_k^2}{a_{ik}\bar{a}_{jk}-R_k^2} \tag{11}$$

is positive definite. Indeed, $\alpha_{ij} = r_k^2/R_k^2$ when $i=k$ or $j=k$, and

$$\begin{aligned}
\alpha_{ij}-1 &= \frac{R_k^2-r_k^2}{a_{ik}\bar{a}_{jk}-R_k^2} \\
&= (R_k^2-r_k^2)\sum_{m=0}^{\infty}\frac{1}{a_{ik}\bar{a}_{jk}}\left(\frac{R_k^2}{a_{ik}\bar{a}_{jk}}\right)^m
\end{aligned}$$

otherwise. Thus

$$\begin{aligned}
\sum_{i,j=1}^{n}\alpha_{ij}\xi_i\bar{\xi}_j &= \frac{r_k^2}{R_k^2}|\xi_k|^2 + 2\frac{r_k^2}{R_k^2}\operatorname{Re}X + |X|^2 + \sum_{i,j\neq k}^{n}(\alpha_{ij}-1)\xi_i\bar{\xi}_j \\
&= \frac{r_k^2}{R_k^2}|\xi_k+X|^2 + \frac{R_k^2-r_k^2}{R_k^2}|X|^2 \\
&\quad + (R_k^2-r_k^2)\sum_{m=0}^{\infty}R_k^{2m}\sum_{i,j\neq k}^{n}\xi_i\bar{\xi}_j\left(\frac{1}{a_{ik}\bar{a}_{jk}}\right)^{m+1},
\end{aligned} \tag{12}$$

where $X := \sum_{i=1,i\neq k}^{n}\xi_i$. Hence the last expression in (12) is non-negative for all vectors $\xi \neq \mathbf{0}$.

In order to prove that it is in fact strictly positive we assume the opposite. Since all the terms in the right hand side of (12) are non-negative we conclude that

$$X = \xi_k = 0.$$

Hence there is $p \neq k$ such that $\xi_p \neq 0$. On the other hand we see that

$$\sum_{m=0}^{\infty} R_k^{2m} \sum_{i,j \neq k}^{n} \xi_i \bar{\xi}_j \left(\frac{1}{a_{ik}\bar{a}_{jk}}\right)^{m+1} = \sum_{m=0}^{\infty} R_k^{2m} |\sum_{i \neq k}^{n} \frac{\xi_i}{a_{ik}^{m+1}}|^2$$

whence our assumption yields

$$\sum_{i \neq k}^{n} \frac{\xi_i}{a_{ik}^{m+1}} = 0, \qquad m = 0, 1, 2, \ldots$$

The last system of linear equations together with the characteristic Vandermonde determinant property and the fact that $\xi_p \neq 0$ imply that there are two indices $i \neq j$ distinct from k such that

$$a_{ik} = a_{jk}.$$

But the latter immediately yields $a_i = a_j$ and this contradiction proves that (11) is a positive definite matrix.

By (10) we have $q_{ij} = Q_{ij}\alpha_{ij}$, where $\|Q_{ij}\|$ and $\|\alpha_{ij}\|$ are Hermitian positive definite matrices. Hence the theorem of I. Schur about the Hadamard product yields that (q_{ij}) is positive definite and the proposition is proved completely.

□

4. Factorization of the determinant function

Now we return to the symmetric collections $\mathcal{B} = \mathcal{B}_n(r)$ given in (3). Then the corresponding matrix (1) takes the form

$$Q_{ij}(r) = -\prod_{k=1}^{n} (\epsilon_{ij}^k + 1 - r^2), \tag{13}$$

where

$$\epsilon_{ij}^k = \omega^{i-j} - \omega^{k-j} - \omega^{i-k}.$$

Lemma 1. *Let ρ_n be given by (4). Then for all $n \geq 2$, ρ_n is equal to the smallest positive zero of the determinant function* $\det \|Q_{ij}(r)\|$. *Moreover, ρ_n is the maximal possible in the sense that $\mathcal{B}_n(r)$ is positive if and only if $r \in (0, \rho_n)$.*

Proof. By Proposition 1 $\mathcal{B}_n(r)$ is positive for all $r > 0$ sufficient small. Hence for those values r the corresponding matrices $\|Q_{ij}(r)\|$ have only positive eigenvalues.

On the other hand, the first principal minor of $\|Q_{ij}(r)\|$ (i.e. the first diagonal element $Q_{11}(r)$) changes its sign at $r_k = |a_{1k}| > 0$ for all $k = 2, \ldots, n$. Hence $\|Q_{ij}(r)\|$ can not be positive definite for all $r > 0$.

The latter implies (by Sylvester's criterium and standard continuity argument) that $\det\|Q_{ij}(r)\|$ has a zero in the semi-interval $(0,\min_k\{r_k\}]$.

Denote by α the smallest zero of $\det\|Q_{ij}(r)\|$. By virtue of positivity of $\mathcal{B}_n(r)$ for small r, $\|Q_{ij}(r)\|$ stays positive definite until r reaches α. Hence, by virtue of (4) we have $\rho_n=\alpha$.

In order to prove the last assertion of the lemma, let us assume that $\|Q_{ij}(r)\|$ is positive definite for some $r>\rho_n$. Then property (ii) in Proposition 2 would yield the positive definiteness of $\|Q_{ij}(\alpha)\|$. But the latter contradicts to the definition of α. $\square$

Now we change the notation by setting

$$A_{ij}(z):=-Q_{ij}(\sqrt{1+z})=\prod_{k=1}^{n}(\epsilon_{ij}^k-z),$$

where

$$z=r^2-1\geq -1. \tag{14}$$

Then the corresponding determinant function takes the form

$$\mathcal{A}(z):=\det\|A_{ij}(z)\|_{1\leq i,j\leq n}.$$

Corollary 1. *$\rho_n=\sqrt{1+\zeta_n}$, where ρ_n is given by (4) and ζ_n is the smallest, but not equal to -1, zero of $\mathcal{A}(z)$.*

We will see below that the above matrix has a rather special form which allows us to express its discriminant explicitly. First we recall some standard definitions and facts from linear algebra. A matrix is called *circulant* if each its row is obtained from the previous row by displacing each element, except the last, one position to the right, the last element being displaced to the first position:

$$\begin{aligned} G&=\mathcal{C}(g_1,\dots,g_n)\\ &=\begin{pmatrix} g_1 & g_2 & \cdots & g_n\\ g_n & g_1 & \cdots & g_{n-1}\\ \cdots & \cdots & \cdots & \cdots\\ g_2 & g_3 & \cdots & g_1\end{pmatrix}\end{aligned}$$

or what is the same,

$$g_{ij}=\begin{cases} g_{j+1-i}, & j\geq i,\\ g_{n+j+1-i}, & j<i.\end{cases}$$

The determinant of a circulant matrix admits the following factorization (see [11, p. 80]):

$$\det\mathcal{C}(g_1,\dots,g_n)=\prod_{k=1}^{n}\sum_{j=1}^{n}\omega^{k(j-1)}g_j. \tag{15}$$

Hence the characteristic polynomial of G is

$$\det(G-\lambda I) = \mathcal{C}(g_1-\lambda, g_2 \dots, g_n) = \prod_{k=1}^{n}[-\lambda + \sum_{j=1}^{n} \omega^{k(j-1)} g_j],$$

and the eigenvalues of G are

$$\lambda_k = \sum_{j=1}^{n} \omega^{k(j-1)} g_j. \tag{16}$$

Lemma 2. *Let*

$$T_{n,m}(z) := \sum_{j=1}^{n} \omega^{m(j-1)} A_j(z), \tag{17}$$

where

$$A_j(z) = A_{1,j}(z), \quad j = 1, \dots, n, \tag{18}$$

and $\omega = e^{2\pi \mathrm{i}/n}$. *Then*

$$\mathcal{A}(z) = \prod_{m=1}^{n} T_{n,m}(z). \tag{19}$$

Furthermore the eigenvalues of the A *matrix are exactly the values of the* T*-polynomials at point* z*:*

$$\lambda_k = T_{n,k}(z), \qquad 1 \le k \le n.$$

Proof. By using the identity

$$\epsilon_{i+m,j+m}^{k} = \omega^{i-j} - \omega^{k-j-m} - \omega^{i-k+m} = \epsilon_{i,j}^{k-m},$$

we get

$$A_{i+m,j+m}(z) = \prod_{k=1}^{n}(\epsilon_{i+m,j+m}^{k} - z) = \prod_{k=1}^{n}(\epsilon_{i,j}^{k-m} - z) = A_{ij}(z).$$

This shows that $A(z) = \|A_{ij}(z)\|_{1\le i,j\le n}$ is a circulant matrix.

Furthermore we have $A(z) = \mathcal{C}(A_1(z), A_2(z), \dots, A_n(z))$, where $A_j(z)$ are defined by (18). Applying (15) and (16) we obtain for the determinant

$$\mathcal{A}(z) = \prod_{k=1}^{n} \sum_{j=1}^{n} \omega^{k(j-1)} A_j(z).$$

and for the eigenvalues of $A(z)$

$$\lambda_k = \sum_{j=1}^{n} \omega^{k(j-1)} A_j(z), \quad k = 1, \dots, n,$$

which completes the proof. □

Corollary 2. *The symmetric collection* $\mathcal{B}_n(r)$ *is positive if and only if all the numbers* $T_{n,m}(r^2-1)$ *are negative,* $1 \le m \le n$*. In particular,* $\rho_n^2 - 1$ *is the smallest, greater than* -1*, zero of polynomials* $T_{n,m}(z)$*,* $1 \le m \le n$*.*

Our next step is to express the above T-polynomials in terms of the hypergeometric functions. We recall that the Gauss hypergeometric function is defined by the series

$$F(a,b;c;x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}, \tag{20}$$

where $(a)_0 = 1$, and $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol. Note that in the case when a and b are negative integers, the corresponding hypergeometric function is just a polynomial in x of degree $\min\{-a,-b\}$.

Theorem 2. *Let $n \geq 2$. Then for $1 \leq m \leq n-1$*

$$T_{n,m}(z) = nC_n^m (-z)^{n-m} F\left(-m, m-n; 1-n; -\frac{1}{z}\right), \tag{21}$$

where C_n^m denote the binomial coefficients and

$$T_{n,n}(z) = n((-z)^n - 1). \tag{22}$$

Proof. We have from (18)

$$\begin{aligned} A_j(z) = A_{1j}(z) &= \prod_{k=1}^{n} (\omega^{1-j} - \omega^{k-j} - \omega^{1-k} - z) \\ &= (-1)^n \prod_{k=1}^{n} \omega^{-j-k} (\omega^{2k} + (z\omega^j - \omega)\omega^k + \omega^{j+1}) \\ &= (-1)^n \omega^{n(n+1)/2} \prod_{k=1}^{n} (\omega^{2k} + (z\omega^j - \omega)\omega^k + \omega^{j+1}) \end{aligned}$$

In order to reorganize the last product we consider an auxiliary quadratic polynomial

$$\zeta^2 + (z\omega^j - \omega)\zeta + \omega^{j+1} = (\lambda_j - \zeta)(\mu_j - \zeta). \tag{23}$$

where λ_j and μ_j are the corresponding zeroes. In view of $\omega^{n(n+1)/2} = (-1)^{n-1}$ we obtain

$$A_j(z) = -\prod_{k=1}^{n} (\lambda_j - \omega^k)(\mu_j - \omega^k).$$

Applying

$$\prod_{k=1}^{n} (x - \omega^k) = x^n - 1,$$

and $\lambda_j^n \mu_j^n = \omega^{(j+1)n} = 1$, we arrive at

$$A_j(z) = -(\lambda_j^n - 1)(\mu_j^n - 1) = (\lambda_j^n + \mu_j^n) - 2. \tag{24}$$

The latter expression, as a symmetric function of λ_j and μ_j, may be polynomially expressed in the coefficients of polynomial (23). Namely by the Cardan identity [8] we have

$$x^n + y^n = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} C_{n-k}^k \cdot \alpha^{n-2k} \beta^k,$$

where $\alpha = x+y$ and $\beta = xy$, and $[p]$ stands for the integer part of x. Hence, applying Viète's formulas

$$\alpha = \lambda_j + \mu_j = \omega - z\omega^j, \qquad \beta = \lambda_j \mu_j = \omega^{j+1},$$

we can rewrite (24) as follows:

$$A_j(z) = -2 + \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} C_{n-k}^k \cdot (\omega - z\omega^j)^{n-2k} \omega^{(j+1)k}. \tag{25}$$

On the other hand, for any m

$$\sum_{j=0}^{n-1} \omega^{mj} = n\delta_m, \tag{26}$$

where

$$\delta_m = \begin{cases} 1, & \text{if } \ m \equiv 0 \mod n; \\ 0, & \text{otherwise,} \end{cases}$$

is the Kronecker symbol modulo n. Therefore we have from (17) and (25)

$$\begin{aligned} T_{n,m}(z) &= -2n\delta_m + \sum_{k=0}^{[n/2]} (-1)^k \frac{nC_{n-k}^k}{n-k} \sum_{j=0}^{n-1} (1 - z\omega^j)^{n-2k} \omega^{j(k+m)} \\ &= -2n\delta_m + \sum_{k=0}^{[n/2]} (-1)^k \frac{nC_{n-k}^k}{n-k} \, S_{m,k}, \end{aligned} \tag{27}$$

where

$$\begin{aligned} S_{m,k} &= \sum_{j=0}^{n-1} (1 - z\omega^j)^{n-2k} \omega^{j(k+m)} = \sum_{j=0}^{n-1} \sum_{p=0}^{n-2k} C_{n-2k}^p \omega^{j(k+m)} (-z)^p \omega^{jp} \\ &= \sum_{p=0}^{n-2k} C_{n-2k}^p (-z)^p \sum_{j=0}^{n-1} \omega^{j(k+m+p)}. \end{aligned}$$

Applying (26) we obtain

$$S_{m,k} = n \sum_{p=0}^{n-2k} C_{n-2k}^p (-z)^p \delta_{k+m+p} = n \sum_{q \in \mathbb{Z}} C_{n-2k}^{qn-m-k} (-z)^{nq-m-k}, \tag{28}$$

where $C_i^j = 0$ for $j > i$ and $j < 0$.

For $q \le 0$ we have $C_{n-2k}^{qn-m-k} = 0$. On the other hand, in view of $k \ge 0$ and $m \ge 1$ we have for all $q \ge 3$

$$qn - m - k \ge 3n - m - k > n - 2k,$$

hence $C_{n-2k}^{qn-m-k} = 0$.

Thus the only non-trivial terms in (28) may occur for $q = 1$ and $q = 2$, which yields

$$S_{m,k} = nC_{n-2k}^{n-m-k}(-z)^{n-m-k} + nC_{n-2k}^{2n-m-k}(-z)^{2n-m-k}. \tag{29}$$

The first binomial coefficient in (29) is non-trivial if

$$\begin{cases} n - m - k \ge 0 \\ n - m - k \le n - 2k \end{cases} \quad \Leftrightarrow \quad \begin{cases} k \le m \\ k \le n - m \end{cases}$$

which gives

$$0 \le k \le m \wedge n := \min\{m, n - m\}.$$

A similar analysis of the second binomial coefficient in (29) shows that it is non-trivial only if $0 \le k \le m - n$ which is equivalent to

$$m = n \quad \text{and} \quad k = 0.$$

In order to finish the proof we return to (27). Assume first that $m = n$. Then $m \wedge n = 0$, that is, $S_{n,k}$ is non-zero only for $k = 0$. Applying the above argument we obtain

$$T_{n,n}(z) = -2n + n\left(1 + (-z)^n\right) = n((-z)^n - 1),$$

which proves (22).

Now let m satisfy $1 \le m \le n - 1$. Then the second term in (29) vanishes and the first term is non-trivial only if $0 \le k \le m \wedge n$ which implies

$$\begin{aligned} T_{n,m}(z) &= \sum_{k=0}^{m\wedge n} (-1)^k \frac{n}{n-k} C_{n-k}^k S_{m,k} \\ &= (-z)^{n-m} \sum_{k=0}^{m\wedge n} \frac{n^2}{n-k} C_{n-k}^k C_{n-2k}^{n-m-k} z^{-k}. \end{aligned} \tag{30}$$

After simple reorganizing

$$\frac{n^2}{n-k} C_{n-k}^k C_{n-2k}^{n-m-k} = n^2 \cdot \frac{(n-k-1)!}{k!(m-k)!(n-m-k)!},$$

and using the Pochhammer notation we obtain

$$\frac{n^2}{n-k} C_{n-k}^k C_{n-2k}^{n-m-k} = (-1)^k n C_n^m \frac{(-m)_k (m-n)_k}{(1-n)_k k!},$$

which finally yields, in view of (30),

$$T_{n,m}(z) = C_n^m n(-z)^{n-m} \sum_{k=0}^{m\wedge n} \frac{(-m)_k(m-n)_k}{(1-n)_k k!}(-z)^{-k}$$
$$= C_n^m n(-z)^{n-m} F\left(-m, m-n; 1-n; -\frac{1}{z}\right)$$

and the theorem is proved completely.

□

We complete this section by identifying the T-polynomials with the classical orthogonal polynomials. Recall that the Jacobi polynomials of degree k are defined for two real parameters $\alpha > -1$, $\beta > -1$ by the following formula

$$\mathcal{P}_k^{\alpha,\beta}(z) = \left(\frac{z-1}{2}\right)^k C_{2k+\alpha+\beta}^k F(-k, -k-\alpha; -2k-\alpha-\beta, -\frac{2}{z-1}) \quad (31)$$

(see [7, p. 212]). Within the above restrictions on α and β, these polynomials constitute an orthogonal family on $(-1, 1)$ with respect to the weight function $w(z) = (1-z)^\alpha(1+z)^\beta$, as k runs through $\mathbb{Z}^+$. It is well known that the zeroes of orthogonal polynomials are real, distinct, and lie in the interior of the orthogonality interval $(-1, 1)$.

Nevertheless, for general α and β the mentioned orthogonality property is no longer valid, but the corresponding Jacobi polynomials are still applicable and a part of their properties can be suitably extended to the general case. The corresponding facts needed for the proof of Theorem 1 are summarized in the next section.

Our formula (21) gives for $m \le n-1$

$$T_{n,m}(z) = (-1)^{n-m} \frac{n^2 z^{n-2m}}{n-m} \mathcal{P}_m^{n-2m,-1}(2z+1). \quad (32)$$

Returning to the old variable r by (14), we get the following explicit representation of the determinant function.

Corollary 3. *Let $\|Q_{ij}(r)\|$ be the matrix in (13). Then*

$$\det \|Q_{ij}(r)\| = c_n \left[1-(1-r^2)^n\right] \prod_{m=1}^{n-1} \mathcal{P}_m^{n-2m,-1}(2r^2-1),$$

where $c_n = (-1)^{\frac{(n-1)(n-2)}{2}} n^{2n-1}/(n-1)!$.

5. The distribution of zeroes

Throughout this section we will suppose that $1 \le m \le n-1$ if not stated otherwise. Let us consider the auxiliary polynomials

$$V_{n,m}(\zeta) = \frac{1}{n} C_n^m F(-m, 1-m; 1-n; \zeta).$$

which are obviously of degree exactly $m-1$. Applying the Pfaff transformation [7, p. 47]

$$F(a,b;c;x) = (1-x)^{-a} F\left(a, c-b; c; \frac{x}{1-x}\right)$$

we obtain

$$F(-m, m-n; 1-n; -\frac{1}{z}) = \frac{(1+z)^m}{z^m} F(-m, 1-m; 1-n; -\frac{1}{1+z}),$$

that in view of (21) yields

$$T_{n,m}(z) = (-1)^{n-m} n^2 z^{n-2m} (1+z)^m V_{n,m}\left(\frac{1}{1+z}\right). \tag{33}$$

Lemma 3. *For all* $m = 1, \dots, n-1$

$$V_{n,n-m}(\zeta) = (1-\zeta)^{n-2m} V_{n,m}(\zeta), \tag{34}$$

and

$$V_{n,m-1}(x) = \frac{1}{(n+1-m)(m-1)} L[V_{n,m}], \tag{35}$$

where

$$L[f] := xf'' - (n-1)f'.$$

Proof. The first formula follows easily from the symmetry of the hypergeometric function with respect to permutation of a and b, and the second Pfaff transformation [7, p. 47]:

$$F(a,b;c;x) = (1-x)^{c-a-b} F(c-a, c-b; c; x).$$

In order to prove the recurrence relation, we apply the standard formula

$$\frac{d}{dx}\left(x^{c-1} F(a,b;c;x)\right) = (c-1)x^{c-2} F(a,b;c-1;x),$$

hence

$$\begin{aligned} \frac{d}{dx}\left(x^{-n} V_{n,m}(x)\right) &= -C_n^m x^{-n-1} F(-m, 1-m; -n; x) \\ &= -(n-m+1) x^{-n-1} V_{n+1,m}(x). \end{aligned} \tag{36}$$

We rewrite this formula as $V_{n+1,m} = \partial_{n,m} V_{n,m}$, where

$$\partial_{n,m} f = -\frac{x^{n+1}}{n-m+1} \frac{d}{dx}(x^{-n} f).$$

On the other hand, applying formula for the derivative of the hypergeometric function

$$\frac{d}{dx} F(a,b;c;x) = \frac{ab}{c} F(a+1, b+1; c+1; x),$$

we get

$$V_{n-1,m-1} = -\frac{1}{m-1} \frac{d}{dx} V_{n,m}.$$

Hence,

$$V_{n,m-1} = \partial_{n-1,m-1} V_{n-1,m-1} = -\frac{1}{m-1} \partial_{n-1,m-1}(V'_{n,m}), \tag{37}$$

which is equivalent to (35). The lemma is proved. □

Now we are ready to formulate the main result of this section.

Theorem 3. *Let $n \geq 4$ and*

$$\nu = [n/2].$$

Then $V_{n,m}(x)$ has only real zeroes and

(i) if $2 \leq m \leq \nu$ then all zeroes of $V_{n,m}(x)$ are distinct and contained in the interval $(1, +\infty)$;

(ii) if $\nu + 1 \leq m \leq n - 1$ then $V_{n,m}(x)$ has exactly $n - m - 1$ simple zeroes in the interval $(1; +\infty)$ and $x = 1$ is a zero of multiplicity $2m-n$.

Proof. The proof will be given by induction on the index n. For $n = 4$ we have $\nu = 2$ and

$$V_{4,2} = \frac{3 - 2x}{2}, \qquad V_{4,3} = (x - 1)^2,$$

which easily yields our claim.

Now suppose that the theorem is valid for some $n = N \geq 4$.

First we establish (i) for $n = N + 1$. By the induction hypotheses, for any m such that $2 \leq m \leq [N/2]$, polynomial $V_{N,m}(x)$ has exactly $m - 1$ real distinct zeroes in the interval $(1; +\infty)$. Denote them in the ascending order $\xi_1 < \dots \xi_{m-1}$ and note that $\xi_1 > 1$.

Consider an auxiliary function

$$f(x) = V_{N,m}(x) x^{-N}.$$

Then $f(x)$ has exactly $m-1$ distinct finite zeroes, and since $\deg V_{n,m} = m - 1 < N$,

$$\lim_{x \to +\infty} f(x) = 0.$$

Applying Rolle's theorem we conclude that the derivative $f'(x)$ has at least $m-1$ distinct finite zeroes. On the other hand, by virtue of (36),

$$V_{N+1,m}(x) = \frac{x^{-N-1}}{m - N - 1} f'(x).$$

Since $V_{N+1,m}(x)$ is a polynomial of degree $m - 1$ it has exactly $m - 1$ distinct zeroes. Denote them by $\{\eta_k\}_{1 \leq k \leq m-1}$. Then

$$\xi_1 < \eta_1 < \xi_2 < \ldots < \eta_{m-2} < \xi_{m-1} < \eta_{m-1} < \infty.$$

This proves (i) for all $m \leq [N/2]$, and since $[N/2] = [(N + 1)/2]$ for even N, (i) is proved for even N.

To complete this inductive step we suppose that N is odd. Then $N = 2\nu + 1$, where $[N/2] = \nu$. By induction hypothesis (ii) is valid for

$n = N$ and $m = \nu + 1$. This shows that $V_{N,\nu+1}(x)$ has one zero $x = 1$ of multiplicity $2(\nu + 1) - N = 1$ and additionally it has

$$N - (\nu + 1) - 1 = \nu - 1 = m - 2$$

real distinct zeroes, all in $(1; +\infty)$. Hence $V_{N,\nu+1}(x)$ has $m - 1$ distinct zeroes.

Arguing as above, we conclude that the polynomial $V_{N+1,\nu+1}$ has $m - 1$ simple real zeroes $\{\eta_k\}_{1\le k\le m-1}$ such that

$$1 < \eta_1 < \xi_1 < \ldots < \eta_{m-2} < \xi_{m-2} < \eta_{m-1} < \infty$$

which finishes the proof of (i).

In order to prove (ii) we make use the symmetry property (34). Namely, let $\nu_1 = [(N + 1)/2]$ and take m such that

$$\nu_1 + 1 \le m \le N.$$

Then we have for the complement index $m' = N + 1 - m$:

$$1 \le m' = N + 1 - m \le N - \nu_1.$$

Since N is integer, we have $2\nu_1 \ge N$. Hence

$$1 \le m' \le \nu_1,$$

that is, m' satisfies the hypotheses of item (i) for $n = N + 1$. Next, by virtue of (34)

$$V_{N+1,m}(\zeta) = (1 - \zeta)^{m-m'} V_{N+1,m'}(\zeta). \tag{38}$$

By the first part of our proof, we know that $V_{N+1,m'}(\zeta)$ has exactly $m'-1$ distinct zeroes in $(1, +\infty)$. Hence by virtue of (38), $V_{N+1,m}(\zeta)$ has the same zeroes and additionally it has a zero at $\zeta = 1$ of multiplicity $m - m' = 2m - N - 1$. This proves the inductive step for (ii) and theorem is proved completely.

□

Our next result establishes the collective properties of the zeroes.

Theorem 4. *Let $n \ge 4$ and $2 \le m \le \nu = [n/2]$. Denote by $\{\xi_i\}$ and $\{\eta_j\}$ the zeroes of $V_{n,m}$ and $V_{n,m-1}$ respectively. Then*

$$1 < \xi_1 < \eta_1 < \xi_2 < \ldots < \eta_{m-2} < \xi_{m-1}.$$

Proof. Let $\varphi_m(x) = V_{n,m}(x)$. Then by (37)

$$\varphi_{m-1}(x) = \frac{1}{(n + 1 - m)(m - 1)} L[\varphi_m], \tag{39}$$

where $L[f] = xf'' - (n - 1)f'$. The second derivative $\varphi''_m(x)$ can be eliminated by using the basic hypergeometric equation for $F(a, b; c; x)$:

$$(1 - x)xF'' + (c - (a + b + 1)x)F' - abF = 0.$$

Namely, by virtue of the definition of $\varphi_m = V_{n,m}$ we can write

$$\varphi''_m = \frac{1}{1 - x}[(n + 1 - 2m)x\varphi'_m + m(m - 1)\varphi_m],$$

hence applying the definition of L and (39), we arrive at

$$L[\varphi_m] = -\frac{2(n-m)}{x-1}\frac{d}{dx}(q(x)\varphi_m'(x) + \alpha\varphi_m(x)), \tag{40}$$

where

$$\alpha = \frac{m(m-1)}{2(n-m)} > 0, \qquad q(x) = x - \frac{n-1}{2(n-m)}.$$

Since $\nu = [n/2]$ and $m \le \nu$ we have

$$\frac{n-1}{2(n-m)} \le \frac{n-1}{2(n-\nu)} < 1.$$

Therefore $q(x) > 0$ for all $x \ge 1$.

Thus, we may rewrite (40) as follows

$$L[\varphi_m] = -\frac{2(n-m)}{(x-1)q^{\alpha-1}(x)} \cdot \frac{d}{dx}(q^{\alpha}(x)\varphi_m(x)),$$

so that (35) in our new notation becomes

$$\varphi_{m-1} = M(x) \cdot \frac{d}{dx}(q^{\alpha}(x)\varphi_m(x)),$$

where

$$M(x) = -\frac{2(n-m)}{(n+1-m)(m-1)(x-1)q^{\alpha-1}(x)}.$$

Now the theorem easily follows from Rolle's theorem. □

The following property is a corollary of the previous theorem and symmetry relation (34).

Corollary 4. *Let $n \ge 4$. Then the maximal zero among all polynomials $V_{n,m}$ when m runs between 2 and $n-1$ coincides with the maximal zero of polynomial $V_{n,\nu}$, where $\nu = [n/2]$.*

6. Proof of Theorem 1

The trivial cases $n = 2$ and $n = 3$ are straightforward in view of (19) and (21). Namely, we find $\rho_2 = \sqrt{2}$ and $\rho_3 = 1$.

Now let $n \ge 4$ and denote by E the full set of zeroes of family $\{T_{n,m}(z)\}_{1 \le m \le n}$. Then Corollary 2 reads as

$$\rho_n' := \rho_n^2 - 1 = \min\{E \cap (-1, +\infty)\}.$$

On the other hand, the first statement of Theorem 1 is equivalent to that ρ_n' is the smallest $\ne -1$ zero of the central polynomial $T_{n,n-\nu}(z)$ where $\nu = [n/2]$. So, what we have to do is to prove that *the number ρ_n' is the smallest $\ne -1$ zero of the central polynomial $T_{n,n-\nu}(z)$, where $\nu = [n/2]$* .

First we note by using (20) that for $m = 1$

$$T_{n,1}(z) = n^2(-1)^{n-1}(1+z)z^{n-2}.$$

Hence $0 \in E$ and it follows that $-1 < \rho_n' \leq 0$. Furthermore,

$$T_{n,n}(z) = n((-z)^n - 1),$$

whence $T_{n,n}(\rho_n') \neq 0$.

Therefore ρ_n' can be characterized as the smallest greater than -1 zero of subfamily

$$\{T_{n,m}(z)\}_{1 \leq m \leq n-1},$$

or equivalently,

$$z = (1 + \rho_n')^{-1}$$

is the largest real zero of family $\{V_{n,m}(z)\}_{1 \leq m \leq n-1}$. But by Corollary 4 we know that this maximum is attained for $m = \nu$, hereby becoming the maximal zero of $V_{n,\nu}(z)$. Moreover, the symmetry relation (34) shows that the same holds also for $V_{n,n-\nu}(z)$.

Hence by virtue of (33) we conclude that

$$0 = T_{n,n-\nu}(\rho_n') = T_{n,n-\nu}(\rho_n^2 - 1)$$

which proves the first assertion of Theorem 1.

In order to finish the proof we return to the asymptotic behavior (5). In view of (32) we see that

$$2\rho' + 1 = 2\rho_n^2 - 1$$

is the smallest $\neq -1$ real zero of $\mathcal{P}_\nu^{n-2\nu,-1}(z)$. By using the transformation formula [10, p. 59]

$$\mathcal{P}_k^{\alpha,\beta}(x) = (-1)^k \mathcal{P}_k^{\beta,\alpha}(-x), \tag{41}$$

we obtain for even $n = 2p$

$$\mathcal{P}_p^{0,-1}(z) = (-1)^p \mathcal{P}_p^{-1,0}(-z), \tag{42}$$

and for odd $n = 2p + 1$

$$\mathcal{P}_{p+1}^{-1,-1}(z) = (-1)^p \mathcal{P}_{p+1}^{-1,-1}(-z).$$

Thus, $z = 1 - 2\rho_n^2$ is the largest zero of $\mathcal{P}_{\nu-\sigma}^{-1,\sigma}(z)$, where $\nu = [n/2]$, and

$$\sigma = 2\nu - n = \begin{cases} 0, & n \text{ is even;} \\ -1, & n \text{ is odd.} \end{cases} \tag{43}$$

Now we can apply a Mehler-Heine type formula [10, Theorem 8.1.2]:

Let $\xi_{k,1} > \xi_{k,2} > \dots$ be the zeroes of $\mathcal{P}_k^{\alpha,\beta}(x)$ in $(-1,1)$ in decreasing order (α, β real but not necessarily greater than -1). If we write $\xi_{k,q} = \cos\theta_{k,q}$, $0 < \theta_{k,q} < \pi$, then for a fixed q,

$$\lim_{k \to \infty} k\theta_{k,q} = j_{\alpha,q}, \tag{44}$$

where $j_{\alpha,q}$ is the qth positive zero of $J_\alpha(z)$, and $J_\alpha(z)$ is the Bessel function of order α.

In our notation $q = -1$, so we have

$$\xi_{n,1} = 1 - 2\rho_n^2,$$

where $\{\xi_{n,j}\}$ denotes the sequence of zeroes of $\mathcal{P}^{-1,\sigma}_{\nu-\sigma}(z)$ in the interval $(-1,1)$ encountered in decreasing order. Then we have from (44)

$$\lim_{n\to\infty}(\nu-\sigma)\arccos(1-2\rho_n^2)=j_{-1,1},$$

which in view of (43) is equivalent to

$$\lim_{n\to\infty} n\rho_n = j_{-1,1}.$$

On the other hand, the Bessel function $J_1(x)=-J_{-1}(x)$, so $j_{1,1}=j_{-1,1}$, which yields (5) and completes the proof.

□

7. Two-side estimates for ρ_n

Denote by $x_{n,k}(a,b)$ the sequence of zeroes, in decreasing order, of the Jacobi polynomial $\mathcal{P}^{a,b}_n(z)$. A classical result of A. Markov states that

$$x_{n,k}(a,b)<x_{n,k}(\alpha,\beta),\quad \forall n\in\mathbb{N},\ \forall k=1,\dots,n, \tag{45}$$

if $-1<\alpha<a$ and $b<\beta<1$ ([10, p. 120], see also [1]).

Note that this result is still true in the limit case: $\alpha=-1$ and $\beta<1$. Indeed, for $-1<\alpha<\beta<1$, $\mathcal{P}^{\alpha,\beta}_n(z)$ is a polynomial of degree exact n and its coefficients (in view of (31)) are continuous functions of u,v outside the lines

$$u+v=-n-1,\dots,-2n.$$

Therefore for any k, $1\le k\le n$, functions $x_{n,k}(u,v)$ are continuous everywhere outside these lines. Hence (45) extends by continuity for all $a>\alpha\ge -1$ and $b<\beta\le 1$.

We will also need the extension of the above monotonicity result in the degenerate case due to Stieltjes [9] (see also [2] and [3] for further discussions). Namely, in the *ultraspherical* case $a=b=\lambda-\frac12$ the positive zeroes

$$x_{n,k}(\lambda)=x_{n,k}(\lambda-\frac{1}{2},\lambda-\frac{1}{2}),\quad k=1,\dots,\nu=[n/2]$$

decrease when λ increase.

Now we are ready to formulate the main result of this section.

Theorem 5. *The sequence ρ_n has the following properties:*
(i) it is decreasing for $n\ge 3$;
(ii) for all $n\ge 3$ the lower estimate holds

$$\rho_n\ge \sin\frac{\pi}{2[\frac{n}{2}]}$$

with equality only if $n=3$;
(iii) for all $n\ge 4$

$$\rho_n\le \sin\frac{3\pi}{4[\frac{n+1}{2}]},$$

with equality only if $n=5$.

Proof. Let us apply the Markov result for $a = b = -1/2$ and $\alpha = -1$, $\beta = 0$. In the first case we obtain the Chebyshev polynomials of the first kind

$$\mathcal{P}_n^{-1/2,-1/2}(z) = \frac{(2n)!}{2^{2n} n!^2} \cos n\theta, \qquad z = \cos\theta,$$

so the corresponding zeroes are

$$x_{n,k}(-\frac{1}{2}, -\frac{1}{2}) = \cos\frac{\pi(2k-1)}{2n}.$$

Then it follows from the proof of Theorem 1 and formula (42) that for $n \geq 2$ $z = 1 - 2\rho_{2n}^2$ is the largest zero of $\mathcal{P}_n^{-1,0}(z)$ which is distinct from 1. Since $z = 1$ is a simple zero of $\mathcal{P}_n^{-1,0}(z)$ (see [10, Section 6.7.2]) we have

$$x_{n,1}(-1,0) = 1, \quad x_{n,2}(-1,0) = 1 - 2\rho_{2n}^2, \tag{46}$$

and by virtue of (45)

$$x_{n,2}(-1/2,-1/2) = \cos\frac{3\pi}{2n} < x_{n,2}(-1,0) = 1 - 2\rho_{2n}^2.$$

Thus for $n \geq 2$

$$\rho_{2n} < \sin\frac{3\pi}{4n}. \tag{47}$$

Let now $\lambda_1 = 0$ and $\lambda_2 = -1/2$ in the Stieltjes theorem. Then for all $n \geq 4$

$$x_{n,2}(-1/2) = 1 - 2\rho_{2n-1}^2 > x_{n,2}(0) = \cos\frac{3\pi}{2n},$$

that is,

$$\rho_{2n-1} < \sin\frac{3\pi}{4n}.$$

Notice also that $\rho_5 = \sqrt{2}/2$ so that the previous inequality becomes an equality for $n = 3$. Combining this with (47) we obtain (iii).

By (42), $z = 1 - 2\rho_{2n-1}^2$ is the largest zero of $\mathcal{P}_n^{-1,-1}(z)$ which is distinct from 1. Hence, by repeating the argument similar to that in the beginning (but for $a = b = -1$) we obtain

$$1 - 2\rho_{2n-1}^2 < 1 - 2\rho_{2n}^2.$$

Hence we have for all $n \geq 2$

$$\rho_{2n-1} > \rho_{2n}. \tag{48}$$

We recall the alternation formula [7, p. 210]

$$C_n^k \mathcal{P}_n^{-k,m}(x) = C_{n+m}^k \left(\frac{x-1}{2}\right)^k \mathcal{P}_{n-k}^{k,m}(x). \tag{49}$$

Then for $k = 1, m = -1$ this formula and (41) yields

$$C_n^1 \mathcal{P}_n^{-1,-1}(x) = C_{n-1}^1 \frac{x-1}{2} \mathcal{P}_{n-1}^{1,-1}(x) = (-1)^{n-1} C_{n-1}^1 \frac{x-1}{2} \mathcal{P}_{n-1}^{-1,1}(-x),$$

hence

$$C_n^1\mathcal{P}_n^{-1,-1}(x)=(-1)^{n-1}C_{n-1}^1\frac{x-1}{2}\mathcal{P}_{n-1}^{-1,1}(-x). \tag{50}$$

On the other hand, by using (41) and making the change of variables $x\to -x$ in (50), we see that

$$n\mathcal{P}_n^{-1,-1}(x)=(n-1)\frac{1+x}{2}\mathcal{P}_{n-1}^{-1,1}(x).$$

Hence in our notation we have $x_{n,n}(-1,-1)=-1$, and also for $k=1,\dots,n-1$:

$$x_{n,k}(-1,-1)=x_{n-1,k}(-1,1).$$

Furthermore, applying (45) to $\mathcal{P}_n^{-1,1}(x)$ and $\mathcal{P}_n^{-1,0}(x)$, we obtain

$$x_{n,k}(-1,0)<x_{n,k}(-1,1),$$

and as a consequence

$$x_{n,k}(-1,0)<x_{n,k}(-1,1)=x_{n+1,k}(-1,-1).$$

Substituting $k=2$ into the latter inequality we obtain for all $n\ge 2$

$$x_{n,k}(-1,0)=1-2\rho_{2n}^2<x_{n+1,k}(-1,-1)=1-2\rho_{2n+1}^2,$$

or $\rho_{2n}>\rho_{2n+1}$. Combining this with (48), we conclude that ρ_k is a decreasing sequence for all $k\ge 3$. Since $\rho_2=\sqrt{2}>1=\rho_3$, the statement (i) in the theorem is proved completely.

In order to prove (ii), we apply again (49) with $k=1, m=0$, which together with (41) yields

$$\mathcal{P}_n^{-1,0}(x)=(-1)^n\frac{1-x}{2}\mathcal{P}_{n-1}^{0,1}(-x).$$

Hence we have for the zeroes: $x_1(-1,0)=-1$, and also for $k=1,\dots,n-1$:

$$x_{n,n+1-k}(-1,0)=-x_{n-1,k}(0,1).$$

In particular, by (46)

$$x_{n,2}(-1,0)=1-2\rho_{2n}^2=-x_{n-1,n-1}(0,1).$$

Then applying (45) for $a=b=1/2$ and $\alpha=0,\beta=1$ we obtain

$$x_{n-1,n-1}(1/2,1/2)<x_{n-1,n-1}(0,1)=2\rho_{2n}^2-1. \tag{51}$$

On the other hand,

$$\mathcal{P}_{n-1}^{1/2,1/2}(z)=\frac{(2n)!}{2^{2n-1}n!^2}\frac{\sin n\theta}{\sin\theta},\quad z=\cos\theta$$

(see, for example, formula (4.1.7) in [10]). Hence $x_{n-1,k}(1/2,1/2)=\cos(\pi k/n)$, $k=1,\dots,n-1$. Applying these formulas to (51) we obtain for all $n\ge 2$

$$\rho_{2n}>\cos\frac{(n-1)\pi}{2n}=\sin\frac{\pi}{2n}.$$

Letting $k = 1$, $m = -1$ in (49) and repeating the above argument, we get

$$n\mathcal{P}_n^{-1,-1}(x) = (n+1)\frac{x^2-1}{4}\mathcal{P}_{n-2}^{1,1}(x),$$

which implies $x_{n,2}(-1,-1) = x_{n-2,1}(1,1)$. Therefore by the Stieltjes inequality in the beginning of this section we obtain for all $n \geq 3$

$$x_{n,2}(-1,-1) = x_{n-2,1}(1,1) < x_{n-2,1}(1/2,1/2) = \cos\frac{\pi}{n-1},$$

that is,

$$1 - 2\rho_{2n-1}^2 < \cos\frac{\pi}{n-1}.$$

Hence we have

$$\rho_{2n-1} > \sin\frac{\pi}{2n-2}.$$

Moreover, for $\rho_3 = 1$ so that we have the equality sign in the latter inequality for $n = 2$. Thus (ii) is proved, and the theorem follows. □

Corollary 5. *For all $n \geq 2$ we have*

$$\rho_n > \sin\frac{\pi}{n}$$

In particular, for all $n \geq 2$ the overlapping coefficient β_n satisfies the inequality $\beta_n > 1$.

8. Appendix: Case $n = 3$

Let and define

$$\mathcal{B}(R_1, R_2, R_3) := \{B(\omega, R_1), B(\omega^2, R_2), B(\omega^3, R_3)\}$$

denote the collection of three circles with *arbitrary* radii R_j and centered at the vertices of the right triangle:

$$a_j = \omega^j, \quad j = 1, 2, 3, \quad \omega = e^{2\pi \mathrm{i}/3},$$

Theorem 6. *$\mathcal{B}(R_1, R_2, R_3)$ is positive if and only if*

$$R_1^2 + R_2^2 + R_3^2 < 3. \tag{52}$$

Proof. Define $x_i = R_i^2$ and note that R_j are subject to the condition (6) which is equivalent to $x_j < 3$ in the new notation. Let $\mathbf{Q} := (Q_{ij})_{1\leq i,j\leq 3}$ denote the matrix in (1) and by Δ_i its principal minor of order i. Then

$$\Delta_1 \equiv Q_{11} = x_1(3 - x_2)(3 - x_3)$$

and the second principal minor

$$\Delta_2 = q[(3-p)x_3^2 - x_3(18 + q - 6p) + 9(3-p)],$$

where $p = x_1 + x_2$ and $q = x_1x_2$. The third minor is found by straightforward computation as

$$\frac{x_1^{-1}x_2^{-1}x_3^{-1}\cdot\Delta_3}{27(3 - x_1 - x_2 - x_2)} = 9 + x_1x_2 + x_2x_3 + x_1x_3 - 3(x_1 + x_2 + x_3). \tag{53}$$

Then by Sylvester's inertia law, $\mathcal{B}(x_1, x_2, x_3)$ is positive if and only if $\Delta_j > 0$ for all $j = 1, 2, 3$.

First we prove that (52) is a sufficient condition for positivity. Indeed, by $0 < x_j < 3$ we have $\Delta_1 > 0$. On the other hand, $x_i > 0$ and applying (52) we see

$$3(3 - (x_1 + x_2 + x_3)) + x_1x_2 + x_2x_3 + x_1x_3 > 0,$$

which immediately yields $\Delta_3 > 0$.

In order to prove that $\Delta_2 > 0$ we notice that $0 < p = x_1 + x_2 < 3$ and consider quadratic polynomial

$$f(x_3) := \frac{\Delta_2}{q(3-p)} = x_3^2 - x_3\frac{18 + q - 6p}{3 - p} + 9.$$

We see that Δ_2 and $f(x_3)$ have the same sign. On the other hand, the symmetry point $x_3 = v$ of the parabola $f(x_3)$ is

$$v = \frac{18 + q - 6p}{2(3-p)} = 3 + \frac{4(3-p) + q}{2(3-p)} > 3,$$

hence $f(x_3)$ is decreasing in $(0, 3)$. Therefore $x_3 < 3 - p$ implies

$$f(x_3) > f(3 - p) = p^2 - q = x_1^2 + x_1x_2 + x_2^2 > 0.$$

Thus $\Delta_j > 0$ for all $j = 1, 2, 3$ and positivity of $\mathcal{B}(R_1, R_2, R_3)$ is proved.

Now we assume that $\mathcal{B}(R_1, R_2, R_3)$ is positive. As above, it suffices only to consider the variable $x = (x_1, x_2, x_3)$ ranges in the cube Q: $0 < x_j < 3$ for all $j = 1, 2, 3$.

Let $\varphi(x_1, x_2, x_3)$ denote the polynomial in the right hand side of (53). Since φ is a harmonic polynomial we obtain by the strong minimum principle

$$\varphi(x) > \min_{\partial Q} \varphi, \quad \forall x \in Q. \tag{54}$$

In order to estimate the minimum in the right hand side we denote by G_i^0 and G_i^3 the edges of Q which correspond to the planes $x_i = 0$ and $x_i = 3$ respectively. One can readily check that the following symmetry relation holds

$$\varphi(3 - x_1, 3 - x_2, 3 - x_3) = \varphi(x_1, x_2, x_3).$$

Hence it suffices only to evaluate the minimum on the edges G_i^0. Moreover, by the usual permutation symmetry, it suffices only to consider one edge G_3^0. Then we have $x \in \partial Q$ and $x_3 = 0$, so that

$$\varphi(x_1, x_2, 0) = (3 - x_1)(3 - x_2) \geq 0,$$

which implies $\inf_{\partial Q} \varphi \geq 0$.

Hence by virtue (54) we have $\varphi > 0$ in Q. By (53) we conclude that inside the cube Q, the function $3 - x_1 - x_2 - x_3$ is either zero or it has the same sign as Δ_3. But the latter sign is positive for all values of x corresponding the positivity condition. Hence positiveness of $3 - x_1 - x_2 - x_3$ is proved and theorem follows. $\square$

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